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The flow due to an oscillating piston in a cylindrical tube: a comparison between experiment and a simple entrance flow theory

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The velocity on the axis of a circular tube was measured over a range of distances from a piston reciprocating in simple harmonic motion. These velocities become independent of axial distance sufficiently far from the piston. The method of calculating the developing flow is based on a comparison with steady laminar flow which, in the entry region of a circular tube, approaches the fully developed state exponentially with distance x from the entry. The steady flow is a function of $x\nu/R^2u_0$ where ν is the kinematic viscosity, R is the tube radius and u_0 is the entry velocity. It is shown that within the limits of experimental error, an oscillating flow follows the steady flow development if u_0 is the instantaneous entry velocity and if the characteristic length is changed from R to the oscillating boundary-layer thickness in the established flow.

1. Introduction

1.1. *Previous work*

The experiments reported here were made on the flow in front of a piston reciprocating in a tube of circular cross-section. The establishment length for oscillating incompressible flow considered here is related to the entrance length in steady flow. If a piston moves uniformly down a tube the distance in front of the piston in which the velocity profile changes from being rectangular at the piston to being parabolic when fully developed is the same as the well-known entrance length. The classical work on entrance length, on the experimental side by Nikuradse and on the theoretical side by Schiller, is discussed by Schlichting (1968, ch. XI). Campbell & Slattery (1963) applied a correction to Schiller's analysis and thus obtained agreement between their theory and the several sets of experimental results, including those of Nikuradse, which they quote. Sparrow, Lin & Lundgren (1964) describe previous calculations of the entrance region in steady flow and present a new approach. In terms of their stretched co-ordinate the approach of the velocity to its asymptotic value is exponential. For the tube of circular cross-section the relationship of stretched to actual axial distance is linear for $x\nu/R^2u_0 > 0{\cdot}04$, where x is axial distance from the entry, ν the kinematic viscosity coefficient, R the tube radius and u_0 the mean velocity. It is remarkable that none of the published work in this field presents logarithmic representations of the approach to the established flow. When relative departure from the asymptotic velocity is plotted logarithmically against a linear scale of $x\nu/R^2u_0$ for steady flow the best results (for example, from Campbell & Slattery) lie very closely on the straight line which is equivalent to

$$u/u_0 = 2 - \exp\left[-16{\cdot}6x\nu/R^2u_0\right], \tag{1}$$

where u is the velocity on the axis of the tube and u_0 the mean velocity.

The entrance region in unsteady flow has received some attention. Avula investigated the entrance region in flow started from rest. In one study (Avula 1969*a*), experimentally determined pressures in the entrance region were used in a solution of the equations of motion to determine the velocity. The flow was found to be significantly different from that obtained using the more usual assumption of pressure gradient independent of time. In the other treatment (Avula 1969*b*) a closed form solution of the integral momentum equation is obtained by the method of characteristics for flow impulsively started from rest. The entrance length (beyond which the velocity profile does not change with distance) is found to increase with time. The entrance length for large times was found to approach Schiller's value for steady flow and is thus half the presently accepted value. The manner of the variation with time will be discussed later.

Some experimental and analytical work on the entrance region in pulsating flow has been published. Far from the ends of a tube the velocity profiles in oscillating flow are known and the oscillating velocities can be simply added to the mean flow to obtain the pulsating flow. The theory of oscillating flow in rigid and flexible tubes has been formulated by a number of authors, for example, Womersley (1955), Uchida (1956) and Cerny & Walawender (1966). Experimentally determined velocity profiles are in agreement with this body of theory (see, for example, Linford & Ryan 1965 or Harris, Peer & Wilkinson 1969). The flow far in front of a piston oscillating in a tube is, therefore, known. The way in which this flow develops with distance from the piston has not been completely investigated though some work has been published. Atabek & Chang (1961) obtained an analytical solution for pulsating flow in the entrance region by linearizing the equations of motion by means of the assumption that the velocity of convection occurring in the inertia terms is instantaneously the same for the whole flow and equal to the velocity at the pipe entry where the velocity profile is rectangular. Their solution can only be applied when the mean flux is large enough for there never to be reversed flow. The non-dimensional frequency $\alpha = R(\omega/\nu)^{\frac{1}{2}}$, where ω is the angular frequency, is the basic parameter of oscillating flow. Atabek & Chang give the non-dimensional entrance length $L\nu/R^2U$ as a function of time for $\alpha = 4$ and for the entrance velocity $U(1+0{\cdot}5\cos\omega t)$. The steady flow entrance length which they quote and about which their entrance length oscillates is small, $L\nu/R^2U$ being $0{\cdot}16$ compared with the accepted value of about $0{\cdot}28$. The outstanding feature of Atabek & Chang's results is that whilst the entrance length oscillates in an approximately sinusoidal manner as one would expect, there is a phase lag of about 60° between the length and the piston velocity. It is interesting to note that the phase difference between volume flux and pressure gradient in the fully developed flow is about 65° (see McDonald 1960, p. 83). When, however, the phase difference between the entry flow and the fully developed flow is calculated after the manner described by Gerrard (1971*a*) it is found to be only 15° for $\alpha = 4$. Whether the phase difference predicted by Atabek & Chang is in fact present, or is just a result of approximation in the theory, needs to be determined.

Atabek, Chang & Fingerson (1964) compared measurements of the velocity profiles in the entrance region of pulsating pipe flow with the theory referred to above and found agreement within the experimental error. When scrutinized in the manner we suggest in this paper there is no indication of the large phase lag expected from the work of Atabek & Chang (1961). The experiments of Atabek *et al.* were performed with $\alpha = 5{\cdot}0$, an amplitude of the pulsation about 0·3 times the mean speed and $UR/\nu = 750$, where U is the temporal mean of the cross-sectional mean velocity. A second experimental study was reported by Florio & Mueller (1968). In their experiments the entrance length for the oscillating component of the pulsating flow was rather short. Their results show that the mean flow development in its entrance length is unaffected by the presence of the oscillating component. They were unable to investigate the reverse effect on the developing oscillating flow.

It is worth noting that Atabek *et al.* (1964) seem to be the only authors who mention that there is an uncertainty in determining the origin of the axial co-ordinate in flow entering a tube. They apply a small correction determined from the theoretical results and measurements close to the entry. This uncertainty does not arise when considering flow in front of a piston in a tube.

1.2. *The basis of the present work*

The entrance length in pulsating flow, in which there is never any reverse flow, can be calculated after the manner of Atabek & Chang (1961). This involves rather laborious series summation. There exists no method of calculation for the case when reverse flow occurs and in particular for oscillating flow when the mean volume flux is zero. A consideration of the physics of the entrance region leads us to suggest a simple method of prediction of oscillating entrance length from the known steady flow value.

The basic parameter, which is the non-dimensional frequency, $\alpha = R(\omega/\nu)^{\frac{1}{2}}$, is a measure of the ratio of the radius to the distance through which vorticity diffuses away from the walls in one period of oscillation. At high frequency, when α is large, there is an extensive region in the centre of the fully developed flow in which vorticity is very small, that is, where the velocity is almost independent of radial position. The velocity in this central region lags the mean (= the piston velocity) by a small amount which tends to zero as α tends to zero and to infinity. (Calculated values are 15° at $\alpha = 4$, 11° at $\alpha = 8$, 6° at $\alpha = 14$ and 5° at $\alpha = 16$.) We may describe the flow outside this central region as an oscillating boundary layer. The boundary layer becomes thinner as α increases. The thickness δ is found by calculating the velocity profile (for example, following Uchida 1956) and adopting some definition of boundary-layer thickness such as that distance from the wall beyond which the velocity remains within 1 % of the value on the axis. Using this definition, δ is $0{\cdot}9R$ in steady flow. It is found that in oscillating flow

δ/R oscillates with an amplitude of about 10 % at $\alpha = 4$, increasing to about 20 % at $\alpha = 16$. The mean δ/R is 0·9 at $\alpha = 4$ and about 0·5 at $\alpha = 14$. The frequency of oscillation of δ is twice the piston frequency because velocity profiles 180° different in phase are identical in shape, but opposite in sign. δ is largest at zero piston speed and smallest at maximum piston speed.

In steady flow the entrance region is where vorticity is produced at the wall. In this region vorticity diffuses as it is convected downstream, to fill the whole cross-section of the tube. The characteristic length which determines the entry length is the tube radius. This is borne out by the finding that non-dimensional velocity on the axis is a universal function of $x\nu/R^2U$ for incompressible steady flow without heat transfer. In oscillating flow vorticity diffuses in the entrance region only across the oscillating boundary-layer thickness δ and so the entrance length is smaller than in steady flow. The reduction is expected to be proportional to δ^2/R^2. It will be shown that the factor of proportionality is close to unity. To test this hypothesis results will be plotted logarithmically as a function of $x\nu/R^2U$ for steady flow and as a function of $x\nu/\delta^2U$ for oscillating flow. In oscillating flow we make the pseudosteady assumption that U is the instantaneous cross-sectional mean speed or the piston speed. The phase difference between the piston and the fully developed core flow is small in any case and so it will be difficult to detect any error in this assumption. Differences if encountered would be expected at large accelerations. This would introduce a further parameter which has not been previously considered. Our experiments and those of other workers were made at small accelerations.

1.3. *Application to previous work*

The experimental results of Atabek *et al.* (1964) are indicated graphically by them to have a rather large uncertainty. The logarithmic plot of the departure from the fully developed state exaggerates the errors. We chose, therefore, to plot their theoretical values taken from the same graphs. There remains a double source of error in the plotting of their results and our readings from the published graphs. The resulting logarithmic plot is shown in figure 1.

The value of α was 5 in this work and δ/R is 90 ± 10 %. R has been used rather than δ as the characteristic length, not only because δ/R is almost unity, but also because the velocity considered is a combination of the steady and oscillating components.

We notice that large scatter is evident when the velocity is less than 10 % different from the fully developed value. This is considered to be simply because small errors are increasingly magnified in this method of plotting as the developed flow is approached. The significant conclusion from figure 1 is that the points lie close to the straight line drawn to the entrance length quoted by Atabek & Chang (1961) rather than to the accepted steady flow result which is also indicated. For this reason further experimental results have been obtained for an oscillating flow to which the theoretical treatment of Atabek & Chang cannot be applied and for which δ/R has the convenient value of about 0·5.

We may briefly consider again the calculated results of Avula (1969*b*) concerning flow impulsively started from rest. The entrance length was found to increase with time. The quoted results give values of non-dimensional time, distance along the tube and boundary-layer thickness. When entrance length is non-dimensionalized using the tube radius it increases by a factor of 34 over the times given. Using boundary-layer thickness instead of tube radius to make the entrance length non-dimensional results in a range of entrance lengths which vary only by a factor of 2·5. The entrance length calculated on this theory for infinite time differs by a factor of 2 from the correct value.

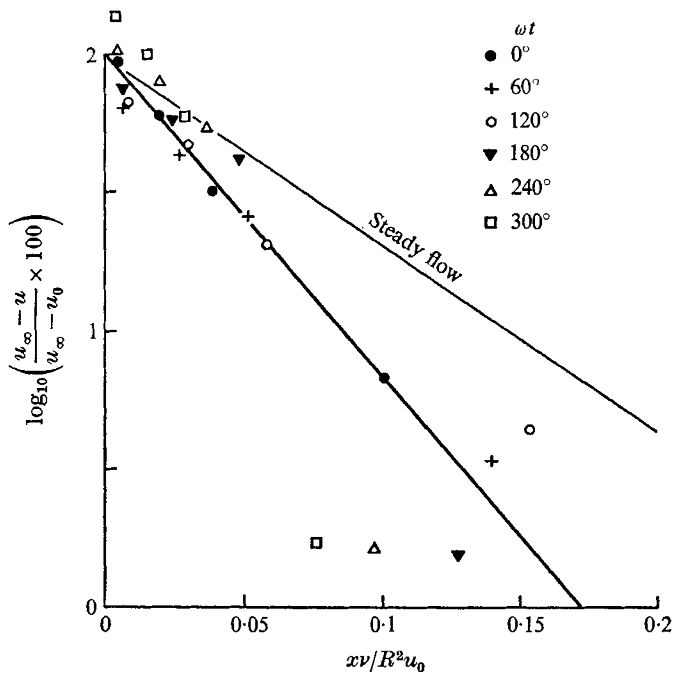

FIGURE 1. Replotted calculations of Atabek *et al.* (1964) for $\alpha = 5$, $\overline{u}_0 R/\nu = 750$. Logarithmic plot of the relative departure of the velocity on the axis from the fully established value as a function of distance from the entry. u_∞ = established velocity on axis, u_0 = entry velocity, u = velocity on axis at distance x from entry, $\overline{u}_0$ = mean entry velocity,

$$u_0 = \overline{u}_0(1 - 0{\cdot}123\cos\omega t + 0{\cdot}321\sin\omega t).$$

2. Apparatus and experimental procedure

The experiments were made in the pulsating flow apparatus previously described by Gerrard (1971*a*). The piston moved in simple harmonic motion at one end of a straight tube 8 m in length. The other end was connected by flexible tubing of the same diameter to a tank open to the atmosphere and situated 4 m above the straight tube. All the measurements were made within a distance of 1 m from the piston. This working section, in which the piston reciprocated, was precision-bore glass tubing 1 m in length and 3·81 ± 0·005 cm in diameter. Glass was used in preference to acrylic tubing because of the variation of inside diameter of plastic tubes. To accommodate the fine wires used in the flow visualization technique the glass tubes were provided with holes at opposite ends of diameters at various distances from the end. In all the measurements the angular frequency of the piston was 0·57 radian s^{-1} and its amplitude was 9·9 cm. The value of $\alpha = R(\omega/\nu)^{\frac{1}{2}}$ was 14·4. The flow was laminar at all times.

The method of measurement of velocities in the entrance region was similar to that employed by Davis & Fox (1967). They used the hydrogen-bubble technique to measure velocities in the entrance region in steady flow. The technique of velocity measurement used in the present work was first described by Baker (1966). The working fluid was a dilute aqueous solution of thymol blue pH indicator which had been titrated to its end point. Increase in alkalinity produces a colour change from yellow to dark blue. A fine platinum wire of diameter 25 μm was stretched across a horizontal diameter of the tube and served as the cathode. The anode was a brass flange at the end of the glass tube. By pulsing a voltage between these electrodes, bands of coloured fluid were produced which drifted with the flow. Velocities were determined by taking cine films of the motion of the bands of coloured fluid. Contrast was improved by illuminating with a sodium discharge lamp. A series of fine wires that moved in phase with the piston were arranged outside the tube so that there was always at least one of them in the field of view of the camera. In this way the phase of the piston at any instant could be calculated from the film.

Only half of the oscillatory motion was filmed: from the fully withdrawn position, 0°, to the fully extended position of the piston during its forward stroke. The voltage was pulsed so that at each of the phase angles under consideration (30°, 60°, 90° and 175°) there was at least one coloured band visible.

The drift of the coloured fluid was measured on the negatives of the film in a measuring machine. Absolute drift distances were obtained by comparison with a calibration photograph of an accurately lined graticule positioned so that its lined surface lay on the horizontal diameters of the fluid filled tube. The graticule was positioned and photographed after each run before the camera was moved. The camera frame speed was about 25 s^{-1} and was accurately determined from photographs of a stop watch. A correction was applied to the velocities determined from the drift distances and the framing interval because the coloured fluid lay in the wake of the wire by which it was produced. The effect is discussed by Schraub *et al.* (1965) and by Davis & Fox (1967) for the case of hydrogen bubbles. Further work which shows that the wake effect is the same in the present method has been completed in this department (Gerrard 1971*b*).

In this method of velocity determination individual measurements of the velocity have a large uncertainty because the edges of the coloured fluid are diffuse. To increase the accuracy many measurements of the same quantity are made on different photographs. Measurements were repeated until there were enough of them to ensure that the velocities they provided were distributed in an approximately Gaussian manner. The values presented below are the arithmetic means and the r.m.s. errors of the means are indicated. These errors are smaller than those of the individual readings in the ratio of $(n-1)^{\frac{1}{2}}$, where n is the number of readings.

3. Presentation and discussion of results

In figure 2 the results are presented as the difference between the fluid velocity on the axis and the piston speed at the four phase angles considered. This quantity is plotted against $x\nu/R^2u_0$ where both x and u_0 are variables. The kinematic viscosity ν and the tube radius R were constant: u_0, the velocity at $x = 0$, is a function of the phase angle only. The dashed curves in figure 2 represent reasonably smooth curves through the points. The expected gradual approach to the fully established condition is evident. Variation close to the piston is more complicated. The results for the phase angle 60° show a peak in the velocity difference close to the piston. This can be explained with reference to the velocity-phase diagram of figure 3. The fully developed fluid velocity lags the piston

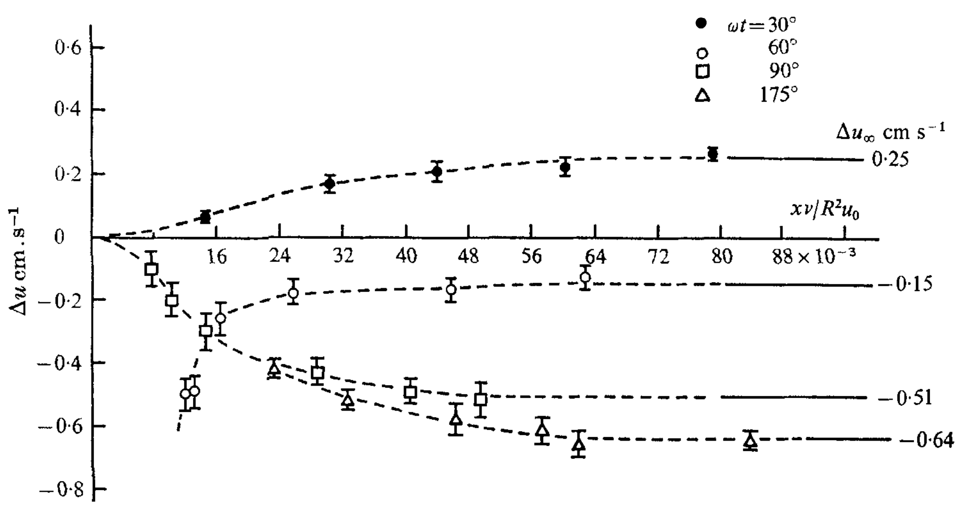

FIGURE 2. Difference between fluid velocity on the axis and piston speed as a function of distance from the piston. $\Delta u = u - u_0$, u = fluid velocity at distance x from piston, u_0 = piston velocity. Bars denote limits of errors. $u_0 \propto \sin \omega t$.

velocity and has a greater amplitude. The fluid very close to the piston will move with the piston speed. The curve of fluid velocity a long way from the piston intersects the piston velocity variation in the region of 60°. Thus the magnitude of the velocity difference Δu, after initially increasing with increasing distance from the piston, will reach a maximum, corresponding to the dashed curves of figure 3, and will then decrease towards the final value. The peak of the curve in figure 2 could not be reached with the wire positions available. A further complication close to the piston is due to the formation of a ring vortex at the start of each forward stroke of the piston. This will be discussed in detail in another publication and is treated also by Tabacynski, Hoult & Keck (1970). At some phase angles, measurements close to the piston were not possible because of the distortion of the velocity field by this vortex.

By using the statistical methods described, reasonable accuracy was obtained. An accurate estimate of the errors involved was also possible. The calculation of the mean and its standard deviation involved between eighteen and thirty readings to ensure a Gaussian distribution of the results. For the phase angles 60° and 90°, at which the velocities were highest, more readings were required. At higher velocities the edges of the dyed fluid were less distinct and so there was a greater uncertainty in the measurements.

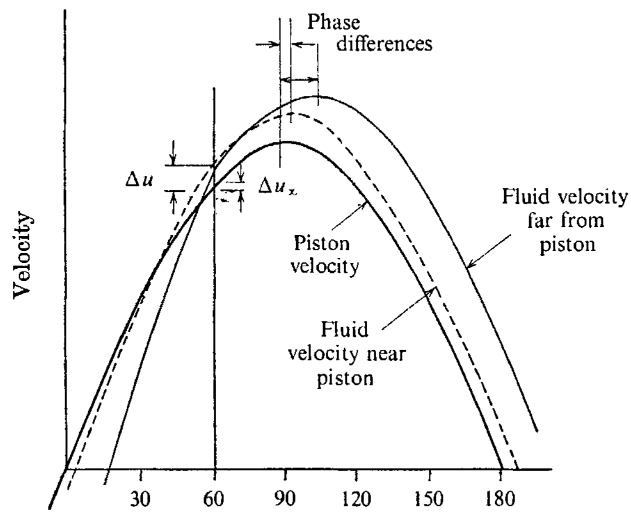

FIGURE 3. Piston and fluid velocities as a function of time and distance from the piston.

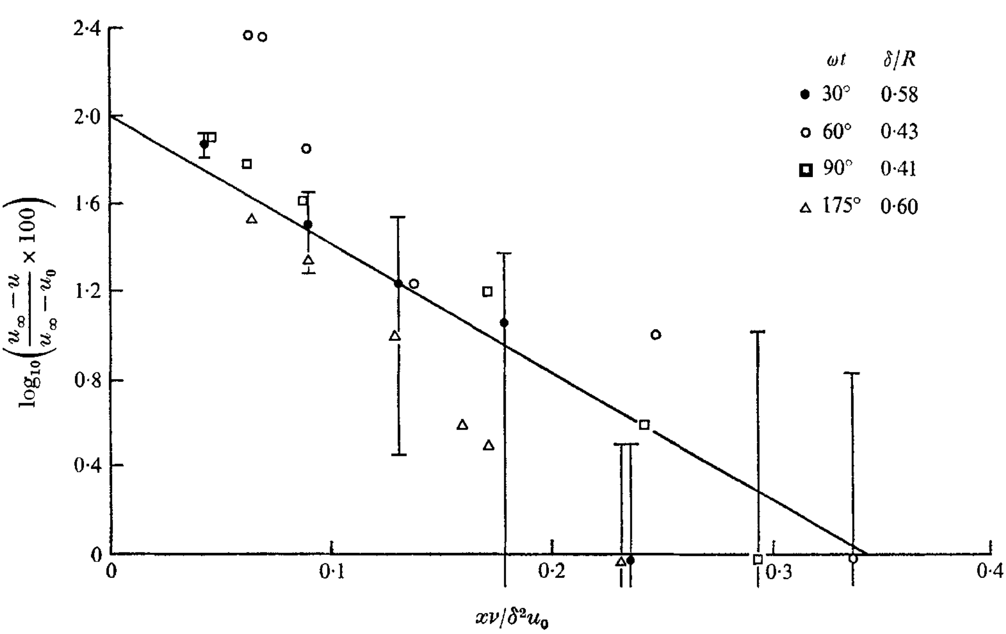

FIGURE 4. Logarithmic plot of the experimental results.
$\alpha = 14{\cdot}4$, $\omega = 0{\cdot}57$ radian s^{-1}, $u_0 = 5{\cdot}64 \sin \omega t$ cm s^{-1}.

The establishment length and the approach to the fully developed flow are displayed in figure 4 which shows the measurements plotted logarithmically against distance from the piston. This distance is non-dimensionalized using the instantaneous piston speed and the oscillating boundary-layer thickness δ. Values of δ/R at the relevant phase angles are shown on the figure. The limits of error shown for one phase angle are typical of all the measurements. These limits increase at small values of the relative departure from the fully developed velocity because of the nature of the logarithm: at the right-hand side of the graph the limits extend to minus infinity. The upper limit of error is shown for all points which have a negative ordinate. Also shown is the approach, on the tube axis, to fully developed steady flow, based on $\delta = 0{\cdot}9R$. The velocity at $0{\cdot}9R$ is 99 % of the axial velocity in steady flow. It is seen that the steady flow curve is a reasonable representation of the oscillating flow behaviour. More precise measurements are obviously needed to discriminate further. Some measurements diverge from the straight line at small distances from the piston for the two reasons explained above.

4. Conclusions

As closely as present measurements show, the development of an oscillating flow in the entry region of a pipe of circular cross-section is the same as it is for steady flow if the axial distance x from the entry is expressed as $x\nu/\delta^2 u_0$, where ν is the kinematic viscosity coefficient, u_0 is the instantaneous entry velocity and δ is the oscillating boundary-layer thickness of the established flow. The definition chosen for δ is that distance from the wall beyond which the velocity differs by less than 1 % from the centre-line velocity. In steady flow this is $0{\cdot}9$ times the tube radius. The conclusions of Florio & Mueller (1968) suggest that the method will also apply to pulsating flow in which the mean volume flux is not zero. Measurements at higher frequency, ω, are needed to determine the effect of acceleration on the pseudo-steady approach.

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